CARTIER'S FIRST THEOREM FOR WITT VECTORS ON $\mathbb{Z}^n_{\geq 0} - 0$

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ABSTRACT. We show that the Witt vectors on $\mathbb{Z}^n_{\geq 0} - 0$ as defined by Angeltveit, Gerhardt, Hill, and Lindenstrauss represent the functor taking a commutative formal group G to the maps of formal schemes $\hat{\mathbb{A}}^n \to G$.

arXiv:1211.1004v1 [math.KT] 5 Nov 2012

1. INTRODUCTION

Hesselholt and Madsen computed the relative K-theory of $k[x]/\langle x^a\rangle$ for k a perfect field of positive characteristic in [HM], and give the answer in terms of the Witt vectors of k. In the analogous computation for the ring $A = k[x_1, \ldots, x_n]/\langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle$, Angeltveit, Gerhardt, Hill, and Lindenstrauss define an n-dimensional version of the Witt vectors, which they use to express the relative K-theory and topological cyclic homology of A [AGHL, Th. 1.1, Th. 1.3].

We show that the additive formal group underlying their Witt vectors on the truncation set $\mathbb{Z}^n_{\geq 0} - 0$, denoted $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}$, represents the functor taking a commutative formal group G to the pointed maps of formal schemes $\hat{\mathbb{A}}^n \to G$ (Theorem 2.1). The case of $n = 1$ is Cartier's first theorem [C] [H, Th. 27.1.14] on the classical Witt vectors. We also show that the additive group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}$ is self dual (Lemma 2.3).

Acknowledgements. This paper is a result of Teena Gerhardt's lovely talk at the Stanford Symposium on Algebraic Topology, and a discussion with Michael Hopkins about it. There are ideas of Hopkins in this paper, and I warmly thank him for them. I also thank Gerhardt, Michael Hill, and Joseph Rabinoff for useful comments. As always, my gratitude and admiration for Gunnar Carlsson are difficult to express.

2. CARTIER'S FIRST THEOREM FOR WITT VECTORS ON $\mathbb{Z}^n_{\geq 0} - 0$

Here is Angeltveit, Gerhardt, Hill, and Lindenstrauss's n-dimensional version of the Witt vectors, defined in Section 2 of [AGHL]: a set $S \subseteq \mathbb{Z}^n_{\geq 0} - \{0\}$ is a *truncation set* if $(kj_1, kj_2, \ldots, kj_n)$ in S for $k \in \mathbb{N} = \mathbb{Z}_{>0}$ implies that $(j_1, j_2, \ldots, j_n)$ is in S. For $\vec{J} = (j_1, \ldots, j_n)$ in $\mathbb{Z}^n_{\geq 0} - \{0\}$, let $\gcd(\vec{J})$ denote the greatest common divisor of the non-zero j_i. Given a ring R and a truncation set S, let the Witt vectors $\mathbb{W}_S(R)$ be the ring with underlying set R^S and

Date: Wednesday, August 8, 2012.
2010 *Mathematics Subject Classification.* Primary 13F35, Secondary 19D55.
Supported by an American Institute of Mathematics five year fellowship.

addition and multiplication defined so that the ghost map

$$\mathbb{W}_S(R) \to R^S$$

that takes $\{r_{\vec{I}} : \vec{I} \in S\}$ to $\{w_{\vec{I}} : \vec{I} \in S\}$ where

$$w_{\vec{I}} = \sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) r_{\vec{J}}^k$$

is a ring homomorphism, where in the above sum, k ranges over $\mathbb{N}$ and $\vec{J}$ is in S. In [AGHL], one requires S to be a subset of $\mathbb{N}^n$, but the same proof that there is a unique functorial way to define such a ring structure [AGHL, Lem 2.1] holds for $S \subseteq \mathbb{Z}^n_{\geq 0} - \{0\}$. Note that

$$\mathbb{W}_S(R) = \prod_{Z \subsetneq \{1,\ldots n\}} \mathbb{W}_{S_Z}(R)$$

where S_Z is defined $S_Z = \{(j_1, \ldots, j_n) \in S : j_i = 0 \text{ if and only if } i \in Z\}$, and that for $S = \mathbb{Z}^n_{\geq 0} - 0$, we have $\mathbb{W}_{S_Z}(R) \cong \mathbb{W}_{\mathbb{N}^m}(R)$ with $m = n - |Z|$.

Let R be a ring. For any truncation set S, the additive group underlying the ring $\mathbb{W}_S(R)$ determines a commutative group scheme and formal group over R.

Let $\widehat{\mathbb{A}}^n = \operatorname{Spf} R[[t_1, t_2, \ldots, t_n]]$ be formal affine n-space andconsider $\widehat{\mathbb{A}}^n$ as a pointed formal scheme, equipped with the point $\operatorname{Spf} R \to \widehat{\mathbb{A}}^n$ corresponding to the ideal $\langle t_1, \ldots t_n\rangle$. Let $\mathrm{Mor}_{\mathrm{fs}}(\widehat{\mathbb{A}}^n, G)$ denote the morphisms of pointed formal schemes over R from $\widehat{\mathbb{A}}^n$ to a pointed formal R-scheme G. The identity of a formal group G gives G the structure of a pointed formal scheme.

For commutative formal groups G_1 and G_2 over R, let $\mathrm{Mor}_{\mathrm{fg}}(G_1, G_2)$ denote the corresponding morphisms.

2.1. *Theorem. — The additive formal group of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ *represents the functor*

$$G \mapsto Mor_{fs}(\widehat{\mathbb{A}}^n, G)$$

from commutative formal groups over R *to groups, i.e. there is a natural identification*

$$Mor_{fg}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R), G) \cong Mor_{fs}(\widehat{\mathbb{A}}^n, G)$$

for commutative formal groups G *over* R.

To prove Theorem 2.1, we first recall that Cartier duality gives a contravariant equivalence between certain topological R-algebras and R-coalgebras [H, Prop 37.2.7]. For such a topological R-algebra (respectively coalgebra) B, let $B^\star$ denote its Cartier dual

$$B^\star = \mathrm{Mor}_R(B, R)$$

where $\mathrm{Mor}_R(B, R)$ denotes the continuous R-module homomorphisms from B to R (respectively the R-module homomorphisms from B to R). Say that an algebra or coalgebra

is *augmented* if it is equipped with a splitting of the unit or counit map. It is straightforward to see that Cartier duality induces an equivalence between augmented topological R-algebras satisfying the conditions of [H, 37.2.4] and augmented R-coalgebras satisfying the conditions of [H, 37.2.5]. Denote the morphisms in the former category by $\mathrm{Mor}_{\text{top alg}}(-,-)$ and the morphisms in the latter category by $\mathrm{Mor}_{\text{coalg}}(-,-)$.

Proof. First assume that the formal group G is affine. Let A denote the functions of G, so A is a Hopf algebra and $G = \operatorname{Spf} A$.

$$\mathrm{Mor}_{\text{fs}}(\widehat{\mathbb{A}}^n, G) = \mathrm{Mor}_{\text{top alg}}(A, R[[t_1, t_2, \ldots, t_n]]).$$

By Cartier duality,

$$\mathrm{Mor}_{\text{top alg}}(A, R[[t_1, t_2, \ldots, t_n]]) = \mathrm{Mor}_{\text{coalg}}(R[[t_1, t_2, \ldots, t_n]]^\star, A^\star).$$

Let F denote the left adjoint to the functor taking a Hopf-algebra (as defined [H, 37.1.7]) to its underlying augmented coalgebra. Since A is a Hopf-algebra, so is $A^\star$. Therefore,

$$\begin{aligned} \mathrm{Mor}_{\text{coalg}}(R[[t_1, t_2, \ldots, t_n]]^\star, A^\star) &= \mathrm{Mor}_{\text{Hopf alg}}(F(R[[t_1, t_2, \ldots, t_n]]^\star), A^\star) \\ = \mathrm{Mor}_{\text{top Hopf alg}}(A, F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star) &= \mathrm{Mor}_{\text{fg}}(\operatorname{Spf} F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star, G), \end{aligned}$$

where $\mathrm{Mor}_{\text{top Hopf alg}}(-,-)$ denotes morphisms of topological Hopf algebras whose underlying topological R-algebra is as before.

By Lemma 2.4 proven below, the formal group $\operatorname{Spf} F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star$ is isomorphic to the formal group associated to the additive group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$, which we also denote by $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

Thus $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ represents the functor $G \mapsto \mathrm{Mor}_{\text{fs}}(\widehat{\mathbb{A}}^n, G)$ restricted to affine commutative formal groups G. Since $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ is an affine formal group, the identity morphism determines an element of $\mathrm{Mor}_{\text{fs}}(\widehat{\mathbb{A}}^n, \mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R))$, which in turn defines a natural transformation

$$\eta : \mathrm{Mor}_{\text{fg}}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R), -) \to \mathrm{Mor}_{\text{fs}}(\widehat{\mathbb{A}}^n, -).$$

For any formal group G, the sets $\mathrm{Mor}_{\text{fg}}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R), G)$ and $\mathrm{Mor}_{\text{fs}}(\widehat{\mathbb{A}}^n, G)$ extend to sheaves on $\operatorname{Spf} R$. Since locally on $\operatorname{Spf} R$, every formal group G is affine, η is a natural isomorphism.

□

2.2. *Lemma. — The group scheme determined by the Hopf algebra* $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ *is isomorphic to the additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

Proof. For notational convenience, given $\vec{I} = (i_1, i_2, \ldots, i_n)$ and $\vec{J} = (j_1, \ldots, j_n)$ in $\mathbb{Z}^n_{\geq 0}$, let $t^{\vec{I}} = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$, and write $\vec{I} \leq \vec{J}$ when $i_k \leq j_k$ for all k.

$R[[t_1, t_2, \ldots, t_n]]^\star$ is a free R-module on the basis $\{b_{\vec{I}} : \vec{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}\}$ where $b_{\vec{I}}$ is dual to $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$. The R-coalgebra structure is given by the comultiplication

$$b_{\vec{I}} \mapsto \sum_{0 \leq \vec{J} \leq \vec{I}} b_{\vec{J}} \otimes b_{\vec{I}-\vec{J}}, \tag{1}$$

and the augmentation $R \to R[[t_1, t_2, \ldots, t_n]]^\star$ sends r to $rb_{\vec{0}}$.

It follows that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the polynomial algebra

$$R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1\rangle$$

with comultiplication equal to the R-algebra morphism determined by (1). Thus, for any R-algebra B

$$\mathrm{Mor}_{\mathrm{alg}}(F(R[[t_1, t_2, \ldots, t_n]]^\star), B)$$

is the group under multiplication of power series in n variables $t_1, t_2, \ldots, t_n$ with leading coefficient 1 and coefficients in B

$$\{1 + \sum_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - \vec{0}} b_{\vec{I}} t^{\vec{I}} : b_{\vec{I}} \in B\}. \tag{2}$$

Any such power series can be written uniquely in the form

$$\prod_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - 0} (1 - a_{\vec{I}} t^{\vec{I}}) \tag{3}$$

with $a_{\vec{I}} \in B$. It follows that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is isomorphic as a Hopf algebra to the polynomial algebra $R[a_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0} - \vec{0}]$ with comultiplication determined by multiplication of power series of the form (3). By the definition of the Witt vectors, it suffices to show that the Witt polynomials $\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a_{\vec{J}}^k$ are primitives for this comultiplication for all $\vec{I}$ in $\mathbb{Z}^n_{\geq 0} - \vec{0}$. To show this, we may assume that R is characteristic 0 since every ring is a quotient of a characteristic 0 ring. Note that

$$\begin{aligned}
\log \prod_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - 0} (1 - a_{\vec{I}} t^{\vec{I}}) &= - \sum_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - \vec{0}} \sum_{k \in \mathbb{N}} \frac{a_{\vec{I}}^k}{k} t^{k\vec{I}} \\
&= - \sum_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - \vec{0}} \sum_{k\vec{J} = \vec{I}} \frac{a_{\vec{J}}^k}{k} t^{\vec{I}} \\
&= \sum_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - \vec{0}} \Big(\sum_{k\vec{J} = \vec{I}} \gcd(\vec{J}) a_{\vec{J}}^k \Big) \frac{-t^{\vec{I}}}{\gcd(\vec{I})}.
\end{aligned}$$

Thus the group under multiplication with elements (3) is isomorphic to the group with elements $\{a_{\vec{I}} \in B\}$ and whose group operation is such that $\big(\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a_{\vec{J}}^k \big)$ is an additive homomorphism, i.e. the Witt polynomials $\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a_{\vec{J}}^k$ are indeed primitives as desired. $\square$

The additive group scheme of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ corresponds to a *graded* Hopf algebra, meaning that there is a grading on the underlying R-module such that the structure maps are maps of graded R-modules. This grading can be defined by giving $a_{\vec{J}}$ as in Lemma 2.2 degree $j_1 + j_2 + \ldots + j_n$. A graded Hopf algebra B whose underlying graded R-module is free and finite rank in each degree has a graded Hopf algebra dual B^* which we define to have mth graded piece $\mathrm{Gr}_m B^* = \mathrm{Hom}_R(\mathrm{Gr}_m B, R)$ and

$$B^* = \oplus_m \mathrm{Gr}_m B^*.$$

Note the difference with the Cartier dual

$$B^\star = \prod_m \mathrm{Gr}_m B^*.$$

Say that a graded Hopf algebra B is *self dual* if there is an isomorphism $B \cong B^*$. An affine group scheme corresponding to a graded Hopf algebra will be called *self dual* if its corresponding graded Hopf algebra is self dual.

2.3. *Lemma. — The graded additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ *is self dual.*

Proof. We give an isomorphism of graded Hopf algebras

$$F(R[[t_1, t_2, \ldots, t_n]]^\star) \cong F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$$

which is equivalent to the claim by Lemma 2.2.

We saw above that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the polynomial algebra

$$R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1\rangle$$

with comultiplication determined by (1). Thus, an R-basis for $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is given by the collection of monomials $b_{\vec{I}_1}^{m_1} b_{\vec{I}_2}^{m_2} b_{\vec{I}_3}^{m_3} \cdots b_{\vec{I}_k}^{m_k}$ in the variables $\{b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0} - 0\}$. Let $\mathcal{C} = \{c_{\vec{I}_1^{m_1}\vec{I}_2^{m_2}\vec{I}_3^{m_3}\ldots\vec{I}_k^{m_k}} : m_j > 0, \vec{I}_j \in \mathbb{Z}^n_{\geq 0} - 0\}$ denote the dual basis of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$.

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$, so $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ etc. For notational convenience, for $\vec{M} = (m_1, m_2, \ldots, m_n)$ in $\mathbb{Z}^n_{\geq 0} - 0$, let $C_{\vec{M}}$ abbreviate $c_{e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n}}$.

Note that

$$\mu(C_{\vec{M}}) = \sum_{0 \leq \vec{J} \leq \vec{M}} C_{\vec{J}} \otimes C_{\vec{M}-\vec{J}}$$

where μ denotes the comultiplication of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$.

Sending $b_{\vec{I}}$ to $C_{\vec{I}}$ thus defines a morphism of Hopf algebras

$$F(R[[t_1, t_2, \ldots, t_n]]^\star) \to F(R[[t_1, t_2, \ldots, t_n]]^\star)^*,$$

and to prove the lemma it suffices to see that the $C_{\vec{I}}$ are free R-algebra generators of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$.

We first show that the $C_{\vec{I}}$ generate $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ as an R-algebra:

An arbitrary element c of $\mathcal{C}$ is of the form $c_{\vec{I}_1\vec{I}_2\cdots\vec{I}_k}$ with the $\vec{I}_j$ not necessarily distinct in $\mathbb{Z}^n_{\geq 0} - 0$. The coordinates of the vectors $\vec{I}_j$ for $j = 1, 2, \ldots, k$ form some finite list of non-negative integers. Let $m(c)$ be the largest number appearing in this list and $N(c)$ be the number of times $m(c)$ appears. For each $j \in \{1, 2, \ldots, k\}$, let $\vec{I'}_j$ be the vector $\vec{I}_j$ with all of its $m(c)$'s replaced by $m(c) - 1$'s. The multiplication on $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ is dual to

$$b_{\vec{I}_1} b_{\vec{I}_2} b_{\vec{I}_3} \cdots b_{\vec{I}_k} \mapsto \prod_{j=1}^{k} \big(\sum_{0 \leq \vec{J} \leq \vec{I}_j} b_{\vec{J}} \otimes b_{\vec{I}_j - \vec{J}} \big).$$

Thus the difference

$$c_{\vec{I'}_1\vec{I'}_2\vec{I'}_3\cdots\vec{I'}_k} C_{\sum_{j=1}^k (\vec{I}_j - \vec{I'}_j)} - c$$

is a sum of terms $c' \in \mathcal{C}$ such that either $m(c') < m(c)$ or $m(c') = m(c)$ and $N(c') < N(c)$. Since all c such that $m(c) = N(c) = 1$ are of the form $C_{\vec{I}}$, we have by induction on $m(c)$ and then $N(c)$ that the $C_{\vec{I}}$ generate $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ as an R-algebra.

We now show that there are no relations among the $C_{\vec{I}}$, i.e. that the distinct monomials $C_{\vec{I}_1} C_{\vec{I}_2} \cdots C_{\vec{I}_k}$ form an R-linearly independent subset of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$:

Fix $\vec{M}$ in $\mathbb{Z}^n_{\geq 0} - 0$. Let $\mathcal{I}$ denote the set of finite sets $\{\vec{I}_1, \vec{I}_2, \ldots, \vec{I}_k\}$ with $\vec{I}_j$ in $\mathbb{Z}^n_{\geq 0} - 0$ and $\sum_{j=1}^k \vec{I}_j = \vec{M}$. For S in $\mathcal{I}$ with $S = \{\vec{I}_1, \vec{I}_2, \ldots, \vec{I}_k\}$, let $C_S = \prod_{j=1}^k C_{\vec{I}_j}$ in $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ and let $c_S = c_{\vec{I}_1\vec{I}_2\cdots\vec{I}_k}$ in $\mathcal{C}$. Note that for all S in $\mathcal{I}$, C_S is in the sub-R-module $\mathcal{F}_{\vec{M}}$ spanned by $\{c_S : S \in \mathcal{I}\}$. By the above, $\{C_S : S \in \mathcal{I}\}$ spans $\mathcal{F}_{\vec{M}}$. Since $\mathcal{F}_{\vec{M}}$ is isomorphic to R^N where N is the (finite) cardinality of $\mathcal{I}$, any spanning set of size N is also a basis [AM, Ch 3 Exercise 15]. In particular $\{C_S : S \in \mathcal{I}\}$ is an R-linearly independent set. Since any monomial in the $C_{\vec{I}}$ is of the form C_S for some $\vec{M}$, it follows that the distinct monomials in the $C_{\vec{I}}$ form a linearly independent set. □

2.4. *Lemma. — The topological Hopf algebra* $F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star$ *is the ring of functions of the formal group associated to the additive group of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}(R)$.

Proof. The Cartier dual $F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star$ of the Hopf algebra $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the product

$$F(R[[t_1, t_2, \ldots, t_n]]^\star)^\star \cong \prod_{m=0}^{\infty} \mathrm{Gr}_m\, F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$$

over m of the mth graded pieces of the graded Hopf algebra dual. By Lemma 2.3,

$$F(R[[t_1, t_2, \ldots, t_n]]^\star)^* \cong F(R[[t_1, t_2, \ldots, t_n]]^\star) \cong R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1 \rangle,$$

with comultiplication determined by (1). So

$$\prod_{m=0}^{\infty} \mathrm{Gr}_m\, F(R[[t_1, t_2, \ldots, t_n]]^\star)^* \cong R[[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]]/\langle b_{\vec{0}} - 1 \rangle,$$

and applying Lemma 2.2 completes the proof.

□

References

[AGHL] Angeltveit, V., Gerhardt, T., Hill, M.A. and Lindenstrauss, A. *On the Algebraic K-Theory of Truncated Polynomial Algebras in Several Variables*, arXiv:1206.0247, 2012.

[AM] Atiyah, M. F. and Macdonald, I. G., *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.,1969.

[C] Cartier, P., *Modules associés à un groupe formel commutatif. Courbes typiques*, C. R. Acad. Sci. Paris Sér. A-B, vol. 265, 1967, A129–A132.

[H] Hazewinkel, M., *Formal groups and applications*, Pure and Applied Mathematics, vol. 78, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York,1978.

[HM] Hesselholt, L. and Madsen, I., *Cyclic polytopes and the* K*-theory of truncated polynomial algebras*, Inventiones Mathematicae, vol. 130, 1997, number 1, pg. 73–97.

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